Order and disorder in the Local Evolutionary Minority Game

E. Burgos,[1,*] Horacio Ceva,[1] and R.P.J. Perazzo[2]

[1] *Departamento de Física, Comisión Nacional de Energía Atómica, Avenida del Libertador 8250, 1429 Buenos Aires, Argentina*

[2] *Instituto Tecnológico de Buenos Aires, Departamento de Investigación y Desarrollo, Avenida Eduardo Madero 399, Buenos Aires, Argentina*

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We study a modification of the Evolutionary Minority Game (EMG) in which agents are placed in the nodes of a regular or a random graph. A neighborhood for each agent can thus be defined and a modification of the usual relaxation dynamics can be made in which each agent updates her decision scheme depending upon the options made in her immediate neighborhood. We name this model the Local Evolutionary Minority Game (LEMG). We report numerical results for the topologies of a ring, a torus and a random graph changing the size of the neighborhood. We focus our discussion in a one dimensional system and perform a detailed comparison of the results obtained from the random relaxation dynamics of the LEMG and from a linear chain of interacting spin-like variables at a finite temperature. We provide a physical interpretation of the surprising result that in the LEMG a better coordination (a lower frustration) is achieved if agents base their actions on local information. We show how the LEMG can be regarded as a model that gradually interpolates between a fully ordered, antiferromagnetic system and a fully disordered system that can be assimilated to a spin glass.

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INTRODUCTION

There are a great number of situations in which a many agent system self organizes by coordinating all the individual actions. A similar situation is found in many particle, physical systems. The word "coordination" that is used in a social or economic context is replaced by "ordering" in the study of condensed systems. Examples of self organization are for instance the growth of a crystalline structure or a transition leading to some specific magnetic phase.

Interesting situations arise in the case of systems in which the optimal configurations for different particles do collide with the each other. This is the case for instance in a spin glass in which many spins interact through a random interaction. In these cases the system is said to display some degree of *frustration*. A case of a multiagent system displaying frustration that has been widely considered in the literature is the Minority Game (MG) [1]. In this model many players have to make a binary choice and the winning option is the one made by the minority. Frustration may in general arise as a consequence of the particular nature of the interactions between the particles as in a spin glass, by the boundary conditions imposed by the geometry or the system, or the rules to win the game as in the case of the MG. The similarities between the MG and spin glass systems has been discussed in great detail by Ref. [2].

The macroscopic signature of frustration is that the many agent system can not accommodate into a single, optimal state in which the energy is a minimum but relaxes instead to one of many, suboptimal configurations that correspond to local minima in the energy landscape. These configurations display what can be called *quenched disorder* i.e. fail to display an ordered pattern and the associated fluctuations do not disappear at zero temperature. In a spin glass the orientation of all the spins do not reach a unique ordering. In the usual relaxation dynamic that is used to model the MG, each player continually modifies her choices and the final ordering of all the agents in terms of their respective choices is far from unique.

The traditional MG ignores any possible spatial proximity among the players. In the present paper we place the agents in a lattice [3], [4] thus associating a neighborhood to each player. We then let each player to make her decisions depending upon the options made by the players of her neighborhood. We next consider a random relaxation dynamics in which each player gradually adjusts her probability of choosing one given alternative of the binary choice. We call this model the Local Evolutionary Minority Game (LEMG). This framework is particularly suitable to study the interplay between frustration and size effects as well as the emergence of ordered patterns through the relaxation process. It also allows a direct comparison with a many spin system in which the size of the neighborhood is assimilated to the range of the spin-spin interaction.

In this paper we present and discuss relevant results of numerical simulations of the LEMG played in one- and two-dimensional systems. We investigate the influence of the value of the size parameter - or what is the same, the range of the interaction - in the emergence of ordered, optimal configurations. We analyze the case of a one dimensional system comparing the LEMG with a model inspired in an antiferromagnetic spin system at a finite

temperature. At the light of this comparison we show that the LEMG can be regarded as a model in which disorder is gradually introduced, the control parameter being the range of the interaction. A fully disordered system that corresponds to the well known EMG then appears to be a particular case in which the range of the interaction is the same as the size of the system. For any smaller value of the size parameter the system can be assimilated to an antiferromagnet. We use this to extend the model to include fluctuations like those produced by a finite temperature. This amounts to study the self organization process implied in the LEMG including agents that change 1their decision with a finite probability. The comparison has the by-product of providing a physical picture to understand why a finite range in the interaction among the agents has the effect of producing a better coordinated configuration [4] with a lower frustration.

THE RULES OF THE GAME

The EMG

We first consider the traditional EMG [5]. This involves N players that make one binary decision (0 or 1). Each player has a probability $p_i; i = 1, 2, \cdots, N$ of choosing, say, 0. Each player receives one point if her decision places her in the minority and loses a point otherwise. We limit ourselves to consider the situation in which the amount of the loss is the same as the amount of the gain. The effect of relaxing this condition has been considered in [6]. When her account of points falls below a given threshold, she changes $p_i \to p_i'$ with $p_i' \in [p_i - \delta p, p_i + \delta p]$, at random, and $\delta p \ll 1$. Reflective boundary conditions are imposed at $p_i = 0, 1$. All agents are assumed to update the corresponding p_i's synchronically.

It is customary to display the self-organization of the system through the probability density function $P(p)$ obtained in a statistical ensemble of systems that are allowed to relax to equilibrium. This function gives the fraction of the population having a probability between p and $p + dp$ of choosing, say, 0. When the probabilistic relaxation is used, the asymptotic function $P(p)$ is shaped as a U with two symmetric peaks at $p \simeq 0$ and $p \simeq 1$ thus indicating that the N agents have segregated into two parties making opposite decisions.

The relaxation process corresponds to the minimization of an "energy" function $\mathcal{E} = \sigma^2/N$ [7] where σ is the standard deviation of the distribution of groups of agents defined by:

$$\sigma^2 = \sum_A \mathcal{P}(A)(A - N/2)^2 \qquad (1)$$

where $\mathcal{P}(A)$ is the probability distribution of groups of A agents that have chosen, say, 0. In [7] it is proven that:

$$\mathcal{E} = \frac{\sigma^2}{N} = N(< p > -1/2)^2 + (< p > - < p^2 >). \qquad (2)$$

One can thus see that the value of σ^2 depends upon the properties of the above mentioned $P(p)$. At equilibrium σ^2 is an extensive magnitude proportional to N. A minimization of $\mathcal{E}$ is equivalent to find a distribution $P(p)$ with the smallest possible number of losers as follows from the fact that $\sigma^2 =< (A - N/2)^2 >=< (w - l)^2 > /4 =< (N - 2l)^2 > /4$ where w (l) is the number of winners (losers). If one assumes naïvely $P(p) = \delta(p - 1/2)$ corresponding to a symmetric random walk (and also eliminating the term $O(N)$ in Eq.(2) one gets $\mathcal{E} = 1/4$ while $P(p) =$ constant yields $\mathcal{E} = 1/6$. A better result is obtained with the usual random relaxation dynamics for the EMG yielding[7] $\mathcal{E} \simeq 1/8$.

Energy and frustration remain linked to each other. For the EMG we can define frustration as $\mathcal{F} = l/N$; which fulfills $0 \leq \mathcal{F} \leq 1$. This definition may also be used for any system involving a game with multiple players. The value $\mathcal{F} = 0$ corresponds to a situation such as the "majority game" in which a player is a winner if her decision is the same as the majority. This leads to situations that can be assimilated to a ferromagnetic phase (all the players (spins) have chosen the same option (orientation)). In the EMG there are less winners than losers, and therefore $1/2 < \mathcal{F}_{EMG} \leq 1$. The lowest possible frustration for the EMG is reached when the N (odd) agents are coordinated to produce the largest possible minority, i.e. $(N - 1)/2$. Thus the lowest possible frustration for a finite EMG is $\mathcal{F}^*_{EMG} = 1/2(1 + 1/N)$.

The LEMG

Within the LEMG a set of neighborhoods $\{\mathcal{N}_i\}$ is constructed, one for each player. All neighborhoods have the same (odd) number n of agents. We consider that the $i - th$ agent belongs to $\mathcal{N}_i$. We define a size parameter by $\zeta = n/N$. We define the set of neighborhoods assuming two possible spatial orderings that correspond respectively to a one-dimensional chain (1D) or a square two-dimensional (2D) regular grid, both with periodic boundary conditions (i.e. corresponding respectively to a ring or a torus). When the agents are placed in a regular lattice the set $\{\mathcal{N}_i\}$ can be generated as a sliding (linear or square) window of size n.

The rules of the LEMG are the same as for the EMG except for the important difference that an agent wins or loses points depending whether she is, or she is not, in the minority *of her own neighborhood*. The $i-$th agent pays no attention whatsoever to the agents that do not belong to $\mathcal{N}_i$. The LEMG contains the EMG as a particular case. In fact the LEMG with $\zeta = 1$ (or $n = N$) is an N-fold replica of the usual EMG when $\mathcal{N}_i$ coincides with

the complete N-agent system. In the regular orderings in which the neighborhoods are respectively a segment or a square with an odd number of agents, the only agent that updates her p_i is located at the center of the square or segment. All agents are assumed to check their respective neighborhoods and update the corresponding values synchronically.

In spite of the fact that each player adjusts her decision according to her respective neighborhoods, we will be interested in computing the energy of the whole system, i.e. we want to check whether the choices made by each agent according to her respective neighborhoods lead the whole ensemble to a more efficient coordination scheme. Notice that in principle there are no reasons to assume any kind of correlation between local and global optima: an agent may be a winner in her neighborhood and a loser when the entire system is considered and vice versa.

RELEVANT FEATURES OF THE LEMG

The density distributions $P(p)$

The density distribution functions $P(p)$ that are obtained with the LEMG for 1D systems are shown in Fig.1. The results shown are an average over 200 samples, each one involving 5×10^5 time steps. Results for 2D systems are completely similar. The main noticeable difference of these functions with those obtained without the neighborhood structure is that they drop essentially to zero in an interval that is symmetric around $p = 1/2$.

As ζ grows, the shape of the corresponding $P_\zeta(p)$ changes. For $\zeta \simeq 1/2$ the distribution has radically changed into the peculiar "two winged" shape [8] that can be seen in the figures, keeping unchanged the fact that $P_\zeta(p \simeq 1/2) = 0$. This is associated to the rapid stabilization of a noticeable majority of one kind of agents in most neighborhoods.

When this happens the agents are naturally induced to take repeatedly the same winning option and keep accumulating points stopping the self- segregation process that is typical of the MG. The many agent system gets "frozen" (Ref. [9]) in a configuration that is far from the "better" *local* optimum. A suboptimal configuration of this kind can be improved resorting to an annealing procedure (see Ref.[9]) that amounts to periodically remove the points accumulated by all the players resetting their accounts to 0. In Fig.1 the annealing procedure is applied every 500 time steps during the first 4×10^5 updatings. It can be seen that the annealing procedure is irrelevant for values of ζ of the order of $\simeq 0$ and $\simeq 1$. The distribution functions obtained after the application of such annealing procedure regain the well known U-shape that displays the self-segregation of the ensemble of agents into two subpopulations with $p_i \simeq 0$ and $p_i \simeq 1$ as a consequence of the relaxation dynamics.

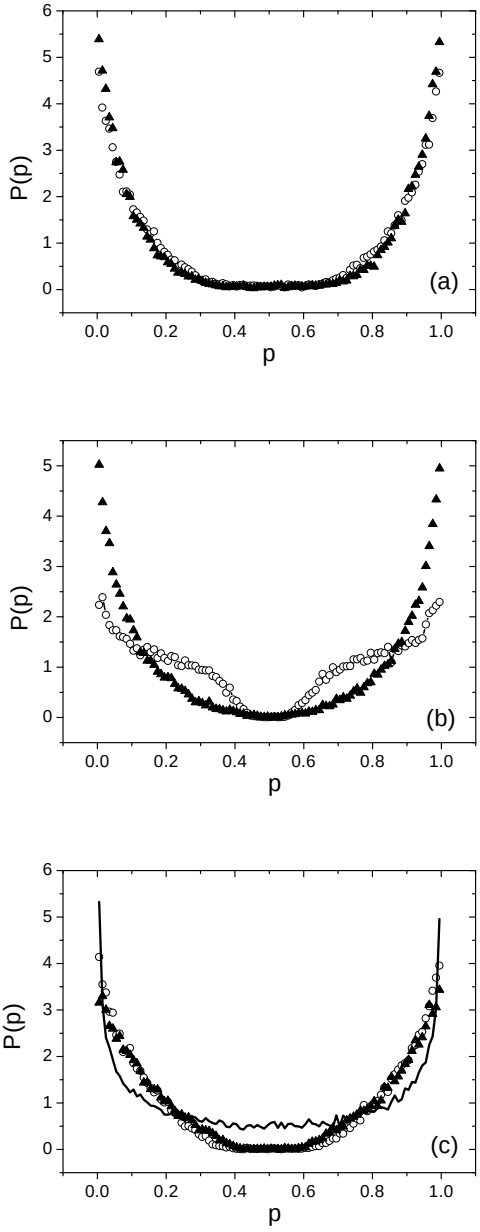

FIG. 1: Density distribution function for a 1D system with N=121 agents, and ζ=0.091 (a), 0.587 (b) and 0.950 (c). Open circles correspond to ordinary relaxation, while full triangles correspond to an annealed relaxation. The full line in (c) corresponds to the EMG, and it is shown for comparison.

The distributions obtained within the LEMG with or without annealing fail to reveal the neighborhood structure of the LEMG. They are unrelated to the underlying neighborhood structure, do not signal the emergence of a long range order and therefore fail to provide any picture of the changes in the emergent ordering of the system as the size parameter changes.

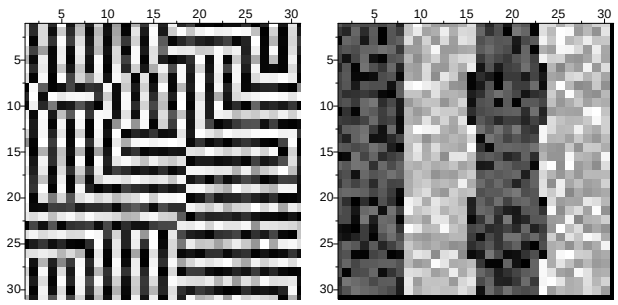

FIG. 2: Results of the relaxation dynamics for a 2D system of 961 agents located in a grid of 31×31 sites after 10^6 time steps. The values of the p_i are shown in shades of gray. Left panel: neighbourhood of 3×3 sites. Rigth panel: neigbourhood of 29×29 sites. Domains are clearly noticeable in the left panel, while on the rigth panel the whole system is in a single domain

The emergence of a long range order

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In Fig.2 we present the results obtained for the case in which the agents are placed in a two dimensional regular lattice. We assume $N = K^2$ (K odd) and $n = k^2$ (k odd). We show the values of the p_i's at the end of the relaxation process by shades of gray. In all cases a pattern arises in which a long range order prevails that depends upon the type and the size of the neighborhood that has been chosen. If the neighborhood is assumed to contain only the four players that are N, S, E and W of the player updating her p_i, a regular checkered pattern emerges that may also alternate with a pattern of "stairs". If a complete neighborhood of eight neighbors is chosen the associated ordered pattern is one of vertical or horizontal stripes. For some values of k ($k \ll K$) the stripes have indented edges.

These regular patterns are in turn grouped into domains with rather sharp boundaries. These "dislocations" are a signature of conflicting boundary conditions (if K is odd the square with periodic boundary conditions can not accommodate an even number of stripes). Most of the agents that are "local losers" - and therefore continue to update their p_i's - lie precisely on these borders giving therefore rise to a dynamics in which the domain borders slowly move.

As the size parameter grows the whole grid becomes a single domain with wider stripes and the domain structure naturally disappears. Beyond a critical value of ζ the whole grid merges into a single domain. As ζ becomes even larger the edges of the stripes have noticeable indentations. Finally these dominate the whole picture and when $\zeta \simeq 1$ all traces of a regular ordering completely disappear.

Domains are a peculiar feature of 2D and higher dimensional system. They do not appear in 1D chains. In Fig 3 we show some typical results for a 1D system. In 1D systems one can not find a domain structure in which portions of the chain in which one periodicity prevails are side by side with another portion in which the prevailing period is different. For larger values of ζ strings of consecutive players with nearly opposite, extreme values of p_i, all having lengths that are very close to a given length, alternate with each other randomly. In addition, this privileged length is close to $n/3$.

It is interesting to note that the annealing procedure mentioned in the preceding section does not change in any way the long range ordering of the system. As can be seen in Fig.3 an improvement of the optimum is achieved only by forcing the agents that already have $p_i > 1/2$ or $p_i < 1/2$ to have p_i's that are closer respectively either to 1 or to 0. As ζ approaches 1 the long range ordering is seen to brake down: strings of consecutive players with opposite values of p change their lengths randomly.

Energy minima

We now consider the total energy $\mathcal{E} = \sigma^2/N$. In Fig.4 we show results for $\mathcal{E}$ as a function of ζ obtained in several numerical experiments. The value for $\zeta = 1$, labeled $\mathcal{E}_{EMG}$, is the one corresponding to the EMG. We show the results of the topologies of a ring and a random system. Data was obtained from 20 independent samples, each one of 5×10^5 time steps, by averaging over the last 2000 time steps of all the samples. The annealing procedure was applied as explained before. In a preceding paper [4] we have pointed out the fact that the total energy and hence the degree of coordination of the N agents is always better for the LEMG than for the EMG provided that $\zeta \simeq 1$ or $\zeta \simeq 0$. In addition if the annealing procedure is followed it turns out that $\mathcal{E}_\zeta < \mathcal{E}_{EMG}$ regardless the value of ζ.

Quenched suboptimal configurations can be improved through annealing, making it possible to reach a"better" local configuration. The composition of the "good" local optima that are obtained in this way always yields values of $\mathcal{E}_\zeta$ that are significantly lower that $\mathcal{E}_{EMG}$. A typical value of $\mathcal{E}_\zeta \simeq 1/16$ is obtained in this way that is half the value $\sigma^2/N \simeq 1/8$ obtained for the stochastic relaxation dynamic. This remarkable results implies that when agents adjust their behavior taking into consideration only the agents of their immediate neighborhood (i.e. having a limited information about the system) they reach a better coordination than when each player considers the behavior of the total ensemble of players (i.e. has a better information about the system as a whole).

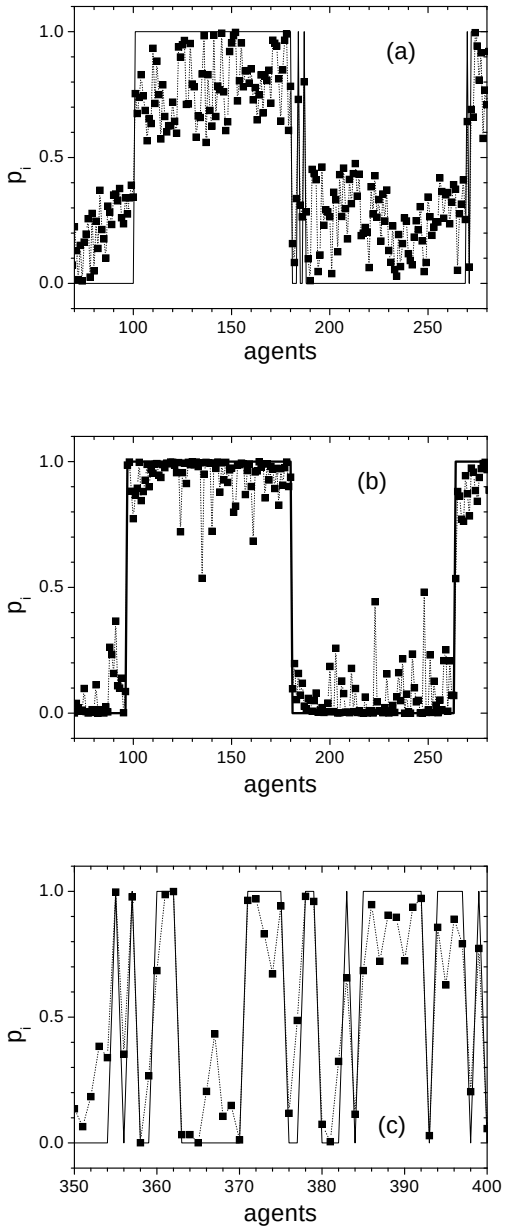

FIG. 3: Results of the relaxation dynamics for a 1D system of 501 agents, after 5×10^5 time steps. We plot the individual probabilities p_i of the agents along the line (full squares). For clarity we show only a small, representative portion of the system. The doted lines are a guide to the eye. The full line is the result of rounding the value of p_i to zero or one. Panels (a) and (b) correspond to a neighborhood of 251 agents, while panel (c) corresponds to one of 501 agents, *i.e.* the EMG. Panel (b) is the annealed version of panel (a). Notice the periodicity displayed in (a) and (b), while the EMG result shows no long range order

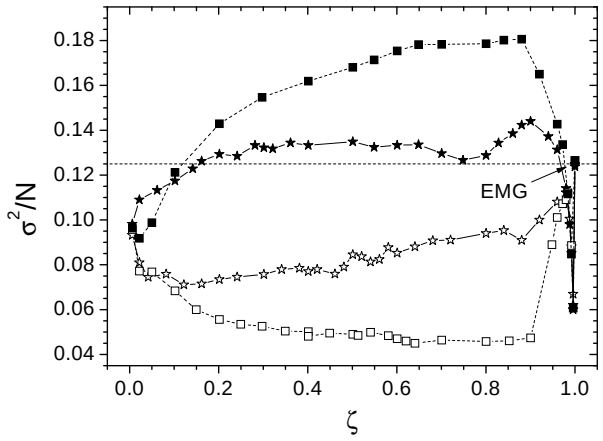

FIG. 4: σ^2/N as a function of the size parameter ζ for different topologies of the system of $N = 501$ agents. Lines are drawn to guide the eye. Stars and squares correspond to 1D and random graphs, respectively. Empty and filled symbols correspond to results obtained with and without annealing (see the text).

THE 1D CASE

The salient features of the LEMG that we have reported in the preceding section can be summarized as follows. The stationary configuration that is reached at the end of the relaxation process, displays a long range order that is related to the size of the neighborhood n. A better understanding is desirable about that relationship as well as the collapse of the internal order as $\zeta \to 1$. Finally it is necessary to supplement the distribution function $P(p)$ with a more reliable indication of that internal order. The density function $P(p)$ clearly displays the self-segregation of the population but is not sensitive to the structure of neighborhoods. A similar statement can also be made about the energy $\mathcal{E}$ introduced in Eq.(2). In spite of the fact that this function turns out to be sensitive to the neighborhood structure, $\mathcal{E}$ is not an explicit function of it. In order to clarify these points we will focus our discussions on the 1D case and we will explore many similarities between the LEMG and a system of interacting spins inspired in an antiferromagnetic Ising model. This can be tailored to be more sensitive to the structure of neighborhoods and also allows a thermodynamic treatment through the introduction of a temperature and a free energy.

The periodic patterns

Let us consider an infinite linear chain of agents and a neighborhood with $n = 3$. Let R_i be the probability that the i-th agent belongs to the minority of $\mathcal{N}_i$. We

can thus write, for $i = -\infty, \cdots, \infty$:

$$R_i = (1 - p_{i-1})p_i(1 - p_{i+1}) + p_{i-1}(1 - p_i)p_{i+1} \quad (3)$$

The probability that all agents are winners is

$$R = \prod_i R_i. \quad (4)$$

Obviously $R = 1$ if and only if $R_i = 1, \forall i$. This is possible only in the case in which $p_i = 1$ and $p_{i\pm1} = 0$ (the opposite case in which $p_i = 0$ and $p_{i\pm1} = 1$ is completely equivalent). This situation corresponds to an ordered pattern in which 0's and the 1's alternate with each other. This is a case in which the alternating strings are of unit length, that in turn corresponds to $n/3$. The case of an infinite linear chain is therefore not frustrated but any finite linear chain with periodic boundary conditions and with an odd number of agents is frustrated.

It is possible to use Eq.(3) to construct a (discrete time) relaxation dynamics to adjust the p_i's. If we assume that $p_i(t + 1) = p_i(t) + \delta p_i$ we can write

$$R_i(t + 1) \simeq R_i + \frac{\partial R_i}{\partial p_i}\delta p_i \quad (5)$$

we can therefore insure that $R_i(t + 1) - R_i(t) \geq 0$ by choosing

$$\delta p_i = \eta \frac{\partial R_i}{\partial p_i} = \eta(1 - p_{i-1} - p_{i+1}) \quad ; \quad \eta > 0 \quad (6)$$

A possible stationary solution ($\delta p_i = 0\ \forall i$) of Eq.(6) is $p_i = 1/2\ ; \forall i$. One can immediately recognize that this solution is unstable because any random, small perturbation of any p_i leads to a dynamics in which $\delta p_i \neq 0; \forall i$. There are other stationary patterns such as the saw tooth profile that repeats the pattern $p_i = 1/2 - \epsilon$, $p_{i+1} = 0$, $p_{i+2} = 1/2 + \epsilon$, $p_{i+3} = 0$, $p_{i+4} = 1/2 - \epsilon$, etc. One can check that these solutions are also unstable. This dynamics stabilizes a pattern of 0's and 1's that alternate with each other. In fact if p_{i+1} and p_{i-1} are both greater (smaller) than $1/2$, then $\delta p_i < 0 (> 0)$ thus forcing $p_i < 1/2 (> 1/2)$. In addition this relaxation dynamics leads to distributions $P(p)$ that vanish at $p = 1/2$ as we have seen in the numerical experiments reported in the preceding section.

Larger (odd) neighborhoods give rise to periodic patterns of alternating strings of 0's and 1's. It is simple to check by inspection that for a neighborhood of length $n = 5$ the length of the strings must be 2. However for $n = 7$, alternating strings of various lengths (1 ,2 and 3) are admissible. The coexistence of several periodic solutions for these rather small values of n is the responsible for the indented edges of the stripes that appear in 2D systems with a neighborhood with $n = 49$ (i.e. a square neighborhood of 7 sites in each direction). It is possible to verify by direct inspection that a concatenation of strings of length $n/3$ tends to insure that players always belong to their local minorities.

For large values of n, the relaxation dynamics leads to asymptotic stationary states in which shorter string lengths do not survive. One can understand this effect on the grounds that a concatenation of short strings is highly unstable against fluctuations in their lengths. When longer strings are involved, a misadjustment by which there are neighboring strings with a length differing in one unit, introduces few losers. A misadjustment of the same kind when short strings are involved causes a larger sequence of losers.

A spin model for the LEMG

The LEMG on a linear chain gives rise to a specific distribution of the lengths of strings of players that make similar decisions. This feature can fruitfully be studied by mapping the LEMG into a spin model related to an Ising antiferromagnetic model. We consider a linear chain of spins $s_i, i = 1, 2, \cdots, N$ with $s_i \in \{-1, +1\}$ and a system of neighborhoods of size n as considered above. In order to compare this model to the LEMG, each spin is assimilated to an agent and its twofold orientation is taken to correspond to the binary option for each agent.

The Hamiltonian function that is a minimum when each agent belongs to the minority of her neighborhood is:

$$H = \frac{1}{4}\sum_i^N s_i \quad \Big[\frac{1}{n-1}\sum_{j\in\mathcal{N}_i; j\neq i} s_j\Big] \quad \equiv \frac{1}{4}\sum_i^N s_i Q_i \quad (7)$$

The factor $(n - 1)^{-1}$ is introduced to have a unified scale of energies independent of the value of the size parameter. The remaining factor $1/4$ is instead introduced in order to render possible the correspondence of H with $\mathcal{E}$ in the limit $\zeta \to 1$. When $n = N$, H can be written as:

$$H = \frac{1}{4(N-1)}\Big[(\sum_i^N s_i)^2 - \sum_i^N s_i^2\Big]. \quad (8)$$

This expression can be cast into the form of Eq.(2) by assuming that $N \gg 1$ and that the discrete spin variables s_i and the continuous probability variables p_i can be associated with each other through [10] $s_i = 2p_i - 1$. Notice that the energy given by Eq.(7) provides an extension of $\mathcal{E}$ with an explicit dependence upon the neighborhood structure. This causes that a low value of H given by Eq. (7), for some given value of n corresponds also to a low value of $\mathcal{E}$ but the reciprocal is not true: one may find a low value of $\mathcal{E}$ in which each player does not belong

to any local minority. This is because $\mathcal{E}$ is associated with the balance within the whole ensemble of players between those that make one choice and those that make the opposite, while H refers instead to the interaction of each player with her neighborhood. The spin-spin interaction in Eq.(7) is not that of a true antiferromagnet because becomes weaker with a larger n. This has been chosen in this way to resemble more closely the rules of the LEMG, in which each agent changes accordingly to the trend in her neighborhood, paying no attention to the actual number of agents in each side.

A random relaxation dynamics that tends to minimize H consists in updating the spins of the system, randomly and asynchronously following the rule:

$$s_i(t) \to s_i(t+1) = -\text{sign}[Q_i(t)] \tag{9}$$

This dynamics leads to asymptotic configurations that are the same as the ones obtained with the LEMG using the random relaxation dynamics in both 1D and 2D systems. In the 2D case it is obtained the same pattern of domains that accommodate horizontal and vertical stripes of widths that are determined by the size n of the neighborhood.

The distribution of string lengths is displayed by the probability P_ℓ of occurrence of a string of length ℓ. In Fig.5 we compare the distribution P_ℓ obtained for the LEMG minimizing H with the spin flipping dynamics of Eq.(9), as the size of the neighborhood is changed. We show as an example the cases with $N = 513$ and n =67, 507 and 513. For $n = 67$ very few string lengths survive and a fully ordered pattern emerges with strings of a length $\ell \simeq n/3 \simeq 23$.

This noisy spectrum in P_ℓ for n=507 is the signature of some degree of disorder of the system. This is, however, a *quenched* disorder because it appears in the absence of any fluctuation that can be attributed to a temperature. In fact as we will immediately see the relaxation dynamics of Eq.(9) can be considered to correspond to the limit $T \to 0$ of a thermal relaxation dynamics. The disorder should therefore be attributed to the frustration imposed by the boundary conditions. In fact it becomes more important when $n/3 > N/4$. *i.e.* when the length of the system is no longer able to accommodate an even number of alternating strings with a length $n/3$. Such disorder dominates when $n = 513$ (i.e. $\zeta = 1$) which corresponds to the EMG; in this case P_ℓ merges into a smooth exponential distribution. This agrees with the well known correspondence between the EMG with a disordered model of a spin glass.

The results displayed above indicate that the LEMG and the spin model provide a similar physical insight. In both models, the size of the neighborhood is a control parameter that allows to gradually interpolate between a fully ordered antiferromagnetic system and a fully disordered, spin glass like system. In the following section we study how the antiferromagnetic model allows a thermodynamical approach that is difficult to envisage as an extension of the EMG.

A THERMODYNAMIC TREATMENT

The occurrence of a pattern of strings of 0's and 1's is a signature of the LEMG that is reproduced within the antiferromagnetic model. The system gets ordered as a result of the minimization produced by the relaxation dynamics. In a general situation also random (thermal) fluctuations should be introduced in this process. These correspond to agents that have a finite probability of changing their decisions at any time. A low temperature can be associated to a very low probability of such change while the high temperature limit should be associated to agents that flip a coin to make up their minds. If thermal fluctuations are allowed, equilibrium corresponds to a minimum of a free energy and this in turn involves an energy minimization and an entropy maximization. In the present section we concentrate in this picture. We will restrict to consider a 1D discrete spin system within the antiferromagnetic model.

The string length distribution

We now discuss how a long range ordering of the system emerges as a consequence of the minimization of a free energy. We focus the discussion of the present section on the case in which $\lim_{N\to\infty} \zeta = 0$. The effects of a finite values of ζ as $N \to \infty$ will be discussed in a forthcoming section within the framework of the spin model presented above.

Let us assume a sequence of N 0's and 1's randomly selected with equal probability. We therefore assume that $N_1 \simeq N_0 \simeq N/2$ up to terms $O(1/N)$. In what follows we shall assume that one kind of characters (say 0's) remains frozen and we will find the distribution of the lengths of strings of the other type (say 1's). Due to the symmetric situation between the 0's and the 1's, what we will conclude for either one is also valid for the other. Let n_m be the number of strings with m equal characters and let M be the total number of these sequences. Consequently the probability of occurrence of a sequence with m characters is $P_m = n_m/M$. In addition holds that $\sum_m m n_m = N/2$. We can also define an average string length through $\hat{\ell} = \sum_m m P_m$.

In order to construct a free energy we have two tasks, one is to define the entropy function and the other is to define the energy. The entropy of a configuration of strings can be calculated as the logarithm of the number of states in which M strings can be arranged preserving the fact that the n_m strings of length m are indistinguishable. Namely

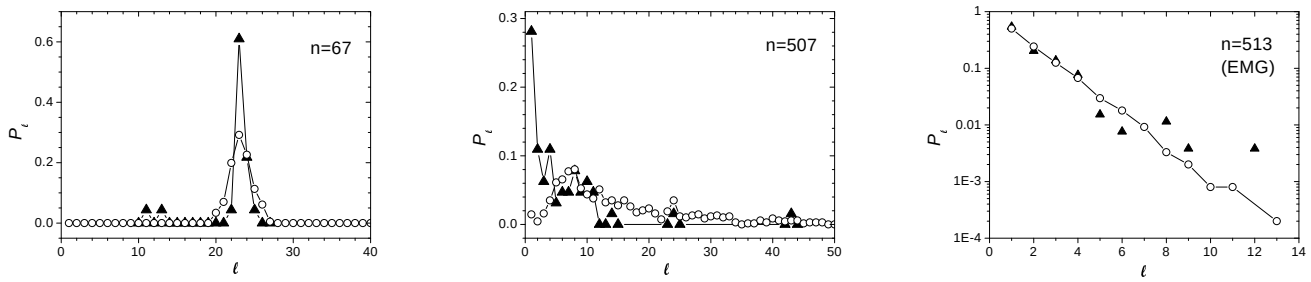

FIG. 5: We show the probability distribution of the string length P_ℓ, for a linear system of 513 agents, for three different neighborhoods. Full triangles correspond to the LEMG, while open circles are the results of the spin model. It can be seen that the n=507 case represents an intermediate situation between that of n=67 (fully ordered), and that of n=513 (fully disordered). Notice that the n=513 case is plotted in a semilog scale.

$$\mathcal{S} = \log\Big[\frac{M!}{\Pi_m n_m!}\Big] \simeq -M\sum_m P_m \log(P_m) \tag{10}$$

The energy function $\widetilde{\mathcal{E}}$ for the same configuration of strings must depend upon the neighborhood structure and therefore must carry the information of the kind of *local* game that are playing the N agents of the system. If an energy function E_m can be defined for a string of m equal characters, the energy of the ensemble of strings can naturally be defined as the weighted sum:

$$\widetilde{\mathcal{E}} = \sum_m P_m \frac{E_m}{\hat{\ell}} \tag{11}$$

We have introduced $\hat{\ell}$ in the denominator in order to work with energies per unit length. We stress the point that $\widetilde{\mathcal{E}}$ is an energy associated to *an ensemble of strings* and is connected to H as given in Eq.(7) only by the spin interaction that is used to calculate the energies E_m. It is clarifying to consider the particular case of $n = 3$. In this case the energy E_m can unambiguously be evaluated with a spin interaction as the one used in Eq.(7). Since the sequence of m equal characters is flanked by one character of the other kind at both ends, E_m can be calculated as in Eq.(7) by sliding a window of width $n = 3$ along the sequence and counting the energy contributed by each position, namely:

$$E_m = \frac{1}{4}\sum_i^m s_m Q_m \tag{12}$$

An alternative possible calculation of E_m is to use a similar sliding window and counting in each site if it corresponds to a local winner or loser. Notice that this procedure is exact only for $n = 3$. For a larger neighborhood additional hipoteses must be made concerning the environment in which the string is placed. We come back to this point later.

We can now turn to find the distribution of the lengths of the strings. For this purpose we need only to evaluate the set $\{P_m\}$ that correspond to a minimum of the free energy per unit length

$$F = \widetilde{\mathcal{E}} - \frac{T\mathcal{S}}{N/2} \tag{13}$$

In order to find a minimum of F, one still has to add a term with the Lagrange multiplier $\lambda(\sum_m P_m - K)$ to impose the condition that $\sum_m P_m = K$ =constant. We have introduced the thermodynamic temperature T that can be associated as usual to fluctuations in the arrangement of strings. We discuss in the next section the microscopic influence of T in a spin model of the LEMG.

In order to calculate the minimum of F we set:

$$\frac{\partial F}{\partial P_\ell} = \frac{\partial \widetilde{\mathcal{E}}}{\partial P_\ell} + \frac{T}{\hat{\ell}^2}\Big[\hat{\ell}(1+\log P_\ell) - \ell\sum_m P_m \log P_m\Big] + \lambda = 0 \tag{14}$$

This equation can be solved for λ. In order to do so we first eliminate the explicit dependence on P_ℓ by multiplying by P_ℓ and performing a summation over ℓ. In this operation one can impose that $\sum P_\ell = K$ but, as expected, the normalization constant K remains undetermined. Solving for λ we obtain:

$$\lambda = -\frac{T}{\hat{\ell}} - \langle\partial\widetilde{\mathcal{E}}\rangle \tag{15}$$

with $\langle\partial\widetilde{\mathcal{E}}\rangle = \sum_m P_m \partial\widetilde{\mathcal{E}}/\partial P_m$. Next λ can be replaced in Eq.(14) to solve for $\log P_\ell$. We finally obtain:

$$P_\ell = \frac{1}{K}\exp[-\ell\mathcal{S}/\hat{\ell}M]\exp[-\hat{\ell}\Delta_\ell/T] \tag{16}$$

where $\Delta_\ell = \partial\widetilde{\mathcal{E}}/\partial P_\ell - \langle\partial\widetilde{\mathcal{E}}\rangle$. Te normalization constant K is determined with the condition that $\sum P_\ell = 1$ because the P_ℓ's are probabilities of mutually excluding events.

The Eq.(16) provides the desired distribution of string lengths. It is interesting to consider the limits $T \to \infty$ and $T \to 0$. The high temperature limit yields a distribution that dies exponentially with the length ℓ of the strings as measured in units of the average length $\hat{\ell}$. This is indeed the distribution found for the (random) distribution used as an initial state for the relaxation process used in the LEMG. As we shall see in the next subsection, this is also found in the high temperature limit of a multi-spin model of the LEMG.

An exponential tail in the distribution is the signature of some kind of disorder. This can either be due to thermal fluctuations as in the present considerations or correspond to a quenched disorder of the system. The latter has been reported in the previous section as the result of the relaxation process when the size of the neighborhood becomes equal to the size of the whole system. Such quenched disorder of the EMG also displays a distribution that has an exponential distribution of string lengths.

To investigate the limit of $T \to 0$ we use Eq.(11) and find that

$$-\frac{\hat{\ell}\Delta_\ell}{T} = -\frac{E_\ell - \widetilde{\mathcal{E}}}{T} = -\sum_m P_m \frac{(E_\ell - E_m)}{T} \qquad (17)$$

This expression allows to find immediately the asymptotic distribution of lengths in the limit $T \to 0$. Assume then that the set of the $\{E_m\}$ is ordered $E_{m_1} < E_{m_2} < \cdots < E_{m_N}$. One can easily recognize that for $T \to 0$, and $\ell = m_N$ the probability $P_\ell \to 0$ because all exponents $-(E_\ell - E_k)/T$ with $k = m_{N-1}, m_{N-2}, \cdots, m_1$ in the second exponential of Eq.(16) are negative causing P_ℓ to approach 0 exponentially for low T. A completely similar argument holds for $\ell = m_{N-1}, m_{N-2}, \cdots m_2$. The only term that survives is $\ell = m_1$ the one corresponding to lowest energy E_{m_1} that, due to the normalization imposed on the probabilities, must fulfill $P_{m_1} \to 1$. We thus see that in the low temperature limit the system gets organized by repeating only strings of a given length $\ell = m_1$ that corresponds to minimum of E_{m_1}. This is what has been obtained both in the simulations reported and within the spin model. As we have seen before the length $\ell = 1$ corresponds to $n = 3$. In general the surviving strings with a minimal E_s are of a length that is close to one third the size of the neighborhood.

Note that an energy E_m can only be defined without ambiguity for the case of $n = 3$. This is so because the local energy of the first and last character of the sequence is completely defined. If the environment is larger, in order to calculate E_m further considerations have to be made about the environment in which the sequence is placed. For larger neighborhoods the energy of longer strings can't be defined. In such a case an average over all possible environments could be made. However, such averaging procedure loses meaning as ζ approaches 1 and the considerations made for the low temperature limit are no longer applicable.

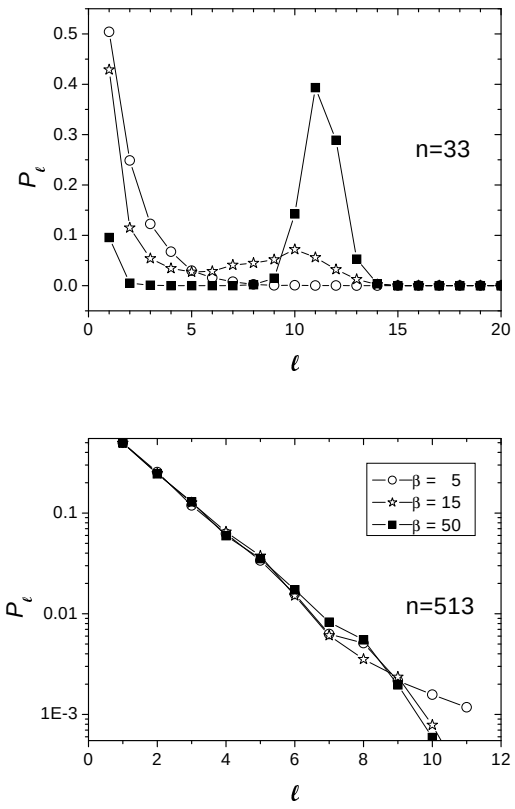

FIG. 6: We show the probability distribution of the string length P_ℓ, for a linear system of 513 agents, for n=33 and n=513 (EMG), and three different temperatures $T = 1/\beta$.

The effects of a finite ζ

Thermal fluctuations in the antiferromagnetic model

A microscopic interpretation of the temperature parameter introduced in the preceding section is desirable. One possibility is to follow the line suggested in Ref.[7]. It has been shown there that the random relaxation used in the LEMG gives rise to fluctuations that can be assimilated to the introduction of a temperature parameter. The fact that the individual probabilities $\{p_i\}$ are updated by choosing the new value at random from an interval $[p_i - \delta p, p_i + \delta p]$ with reflective boundary conditions at $p_i = 1$ and $p_i = 0$ produces fluctuations in the energy function $\mathcal{E}$ given in Eq.(2).

That way is however not acceptable. The introduction of a finite temperature into the LEMG amounts to allow for "hesitating agents" that may change their current decision with a finite probability. The well known gradual updating rule of individual probabilities is un-

able to handle this situation. Within the antiferromagnetic model for the LEMG, such changes can easily be assimilated to a probabilistic flipping of spins in Eq.(9). This temperature parameter is therefore different from the one introduced in [7] because both are associated to a different kind of fluctuations.

To deal with a finite temperature we proceed as usual, namely by assuming that a spin may flip with a probability that depends upon $\beta = 1/T$ through:

$$\text{Prob}(s_i \to -s_i) = \frac{1}{1 + e^{-\beta s_i Q_i}} \tag{18}$$

that contains Eq.(9) in the limit of $\beta \to \infty$ ($T \to 0$).

The ordered pattern for the 1D system in which few string lengths are strongly preferred only holds in the low temperature limit. The fluctuations introduced through the probabilistic flipping of Eq.(18) give rise to a thermal disorder that shows itself through the distribution function P_ℓ. This competes with the quenched disorder produced for large values of the size parameter. As expected, in the high temperature limit $P_\ell \simeq \exp(-\alpha\ell)$.

In Fig.6a we show how the ordered pattern of $n = 33$ melts down as $\beta \to 0$ while for $n = 513$ (Fig.6b) the exponential distribution does not change. The peak in P_ℓ for $n = 33$ that represents an ordered pattern, gradually disappears, being replaced by an exponential for high T. The quenched disorder that appears for $n = 513$ is independent of T.

The local energy function given in Eq.(7) is sensitive to the emergence of a local ordering. The relaxation indicated in Eq.(18) allows to perform a minimization of H while keeping control of the temperature. This allows to investigate the persistence of the local ordering and the competition between quenched and thermal disorder as the size of the neighborhood n approaches the size N of the whole system.

In Fig.7 we show the result of the minimization process for two different values of ζ. As long as $\zeta < 1$ the local energy is lowered as the relaxation proceeds, showing that a local ordering emerges. This reduction gradually becomes less important as $\zeta \to 1$ and vanishes completely in that limit. This puts in evidence that when $\zeta = 1$ the relaxation process is unable to introduce any ordering into the system.

The energy function given in Eq.(7) is expected to show some degree of structure as the size of the neighborhood changes. This is due to the frustration caused by the competing boundary conditions that arise from the concatenation of local minorities in neighborhoods that are comparable to the size of whole chain. Such structure can clearly be seen in the plot of H/N vs ζ that is shown in Fig.8. This pair of variables have been chosen because the corresponding curves turn out to be independent of N. The structure that is displayed must therefore be considered to be independent of the size of the system.

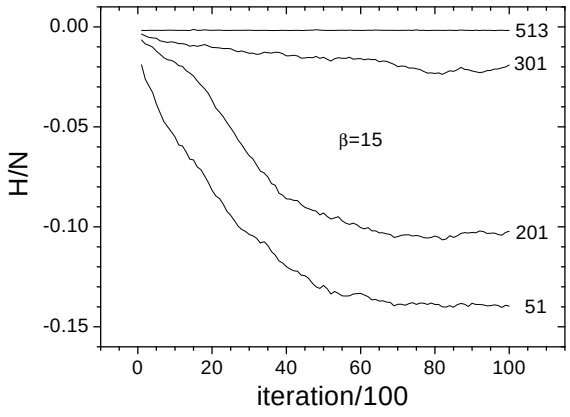

FIG. 7: Evolution of the local energy per site, from the minimization process on the spin model. The number of agents is N=513; The values of n are show in the figure

String lengths that are even fractions of N minimize the frustration because they better fulfill periodic boundary conditions of the whole system by accommodating an even number of strings of equal length along the chain. One therefore expect that the average string length must be related to N through $\hat{\ell} = N/4, N/6, N/8$ etc. On the other hand for an alternating pattern of +1's and -1's, the average string length must in turn related to the size of the neighborhood through $\hat{\ell} = n/3$, in order to achieve a minimal energy. An optimal ordering of the system is therefore possible for $\zeta \simeq 3/4, 3/6, 3/8$, etc. These values are seen to correspond to minima of the curves in Fig.8. The maxima in between correspond to a missmatch between the value of the size of the neighborhood and the size of the whole system, causing that such optimization can not be achieved. This structure is seen not to survive to thermal fluctuations.

CONCLUSIONS

The MG has been greatly used as a working model for self-organization. Individual players, without any central coordination are nevertheless able to minimize a cost function that can be assimilated to an energy, by self-segregating into two different groups that make opposite elections. This features is neatly displayed by the density distribution of individual strategies that shapes as a U.

If the rules of the game are extended by placing the agents into the nodes of a regular or a random graph several important changes occur. In the first place the density distribution of individual strategies turns out to be insensitive to the neighborhood structure of the new model. In the second place a long range spatial ordering dominates the system. In the third place self-

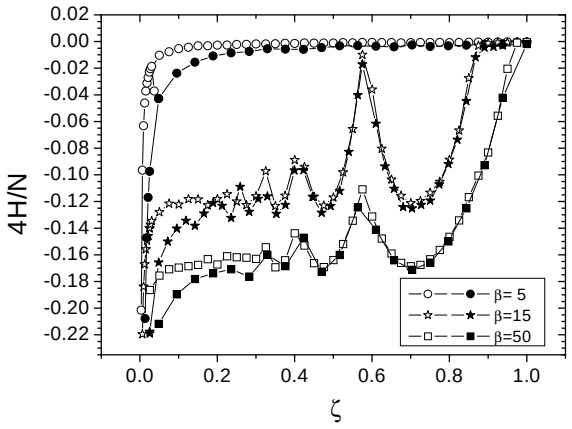

FIG. 8: Energy per site as a function of ζ,for three different values of $T = 1/\beta$. Empty (full) symbols correspond to N=2001 (513) agents.

organization is improved by producing lower value of the energy function.

These features can fruitfully be studied in 1D systems. We found that the distribution of strings of players that make similar options is a relevant parameter that provides a signature of the type and kind of order that is induced in the system. This distribution changes between a delta-like function in which strings of a few lengths are allowed to survive to an exponential distribution of string lengths that is a signature of randomness. This transition puts in evidence that the LEMG provides a gradual interpolation between an ordered system in the limit of small neighborhoods and a fully disordered system when the size of the neighborhoods approaches de size of the system.

We have shown that the LEMG is equivalent to a spin system with an antiferromagnetic-like interaction that extends to a neighborhood whose size is taken as a control parameter. This model displays the same pattern of local ordering and string length distribution and also displays the same order-disorder transition when the range of the antiferromagnetic interaction approaches the size of the whole system. In addition an energy function can be introduced that can be considered as an extension of the LEMG in which the neighborhood structure is explicitly accounted for. A minimum of such energy leads to a minimum of the energy function used for the LEMG but the converse is not true.

The distribution of string lengths can also be studied in the thermodynamic limit of an infinite linear system of spins in a thermal bath. The temperature parameter that has to be introduced is absent in the LEMG and corresponds to hesitating agents that may change their decision with a finite probability at any time. Using this model we were able to find the high and low temperature limits of the probability distribution of string lengths as arising from a minimization of a free energy. The low temperature limit, merges for small neighborhoods into the one derived from the usual relaxation dynamics of the LEMG or the antiferromagnetic model yielding a delta-like distribution of lengths in which only very few privileged lengths survive and a long range ordered pattern emerges.

The high temperature limit can only be treated within the antiferromagnetic model. Within this framework an order-disorder transition can be predicted. Thermal randomness associated to the high temperature limit has the same exponential distribution of string lengths that is numerically obtained in the quenched disordered limit for zero temperature. We have also studied the effect of the interplay of the size of the system and the size of the neighborhood. We have shown that the energy of the spin system as a function of the size parameter can be cast into a form that is independent of the size of the system. This function that is highly structured puts in evidence the occurrence of "preferred" and "hampered" values of the size parameter that correspond to minima and maxima of the frustration introduced by conflicting boundary conditions.

The 1D system that we have considered in the present paper allows to get a detailed physical picture of the LEMG. It helps to understand how a structure of neighborhoods makes it possible a better ordering of the system that is in turn reflected as a minimum of the energy. Such ordered phase that is equivalent to an antiferromagnetic ordering shows up as a better coordination of the decisions made by decentralized agents in a minority game.

* Electronic address: burgos@cnea.gov.ar, ceva@cnea.gov.ar, rperazzo@

[1] D. Challet, Y.C. Zhang, Physica **A246**, 407 (1997); Physica **A256**, 514 (1998). Further references on the Minority Game can be found at: www.unifr.ch/econophysics/minority.

[2] D. Challet, M. Marsili Phys. Rev. **E60**, R6271 (1999); D. Challet, M. Marsili, R. Zecchina, Phys. Rev. Lett. **84**, 1824 (2000).

[3] S. Moelbert, P. De los Rios, Physica **A303**, 217 (2002). Two works where related ideas about local neighborhoods are developed are M. Paczuski, K. E. Bassler, A. Corral, Phys. Rev. Lett. **84**, 3185 (2002), and T. Kalinowski, H-J. Schulz, M. Briese, Physica **A277**, 502 (2000).

[4] E. Burgos, H. Ceva, R. P. J. Perazzo, cond-mat/0212635

[5] N.F. Johnson, P.M. Hui, R. Jonson, T.S. Lo, Phys. Rev. Lett. **82**, 3360 (1999). Very recently the EMG has also been refered to as the 'Genetic Model' (R. Kay and N.F. Johnson, cond-mat/0312556)

[6] E. Burgos, H. Ceva, R. P. J. Perazzo, Phys. Rev. **E64**, 016130 (2001); S. Hod and E. Nakar, Phys. Rev. Lett. **88**, 238702 (2002); E. Nakar, S. Hod, Phys. Rev. **E67**,

016109 (2003); E. Burgos, H. Ceva, R. P. J. Perazzo, Phys. Rev. Lett. **91**, 189801 (2003)
[7] E. Burgos, H. Ceva, R. P. J. Perazzo, Phys. Rev. **E65**, 036711 (2002)
[8] This distribution is actually the composition of two asymmetric distributions, one that vanishes for $p < 1/2$ and the other for $p < 1/2$.
[9] E. Burgos, H. Ceva, R. P. J. Perazzo, Physica **A294**, 539 (2001).
[10] This correspondence is exact only in the limit of a perfect self-segregation.